Supplemental Material I: Step-by-Step derivation

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Abstract—We provide a step-by-step derivation of certain elements of the algorithms and equations provided in the main text of "Collimated Whole Volume Light Scattering in Finite Media". The aim is to aid readers that would like more in-depth explanation of how those relations were derived.

Index Terms—raytracing, color, shading, shadowing, texture.

1 Introduction

The algorithms in the main text are derived following the assumptions explained in Tab. 1. Single scattering might be the focus of the main work, however the sampling algorithm is general enough to be extendable to multiple scattering. Furthermore, it can support gobos with sharp boundaries.

TABLE 1
Table of assumptions

Assumption	Effect
Single-scattering*	Integration is performed by generating samples only along the camera ray.
Homogeneous medium	Extinction, albedo and phase function are constant in the entire medium, allowing analytic integration
Uniform illumination	Allows analytic integration and splitting the integral into an integral over transmittance and emitted radinace by the light source $\mathscr{T}_{\text{box}}(\tilde{\omega}_i, \tilde{\omega}_o, t_S) L_i$.

* Multiple scattering is demonstrated in the main text [1].

2 Deriving a multiple-scattering ratio estimator

The radiative transfer equation in homogeneous medium is expressed in integral form as

$$L(\omega_o,\omega_i) = L_s(\omega_i,\omega_o)\, e^{-\sigma_t t_{SC}} + \int_0^{t_{SC}} e^{-\sigma_t (t+d_m(t))} \sigma_s \int_{\mathbb{S}} f(\omega_i,\omega_o)\, V(t,\omega_i)\, L_i \,\mathrm{d}t. \quad (1)$$

Note, that visibility is combined in L_i. If we treat the integrand as a linear operator with respect to radiance, it can be rewritten as a Neumann series,

$$L' = \sum_{k=0}^{\infty} L_k, \quad (2)$$

where the radiance L_k corresponds to Eq. (1) with L_{k-1} being substituted as L_i, i.e. $L_k = L_{k-1} \circ L$. The unoccluded variant of this equation L'_{u} assumes $V(t,\omega_i) \equiv 1$. Suppose that we have an alternative approach to estimate or compute the unoccluded integral L''_{u}, thus a ratio estimator will perform the following estimate,

$$L_{\text{est}} = L''_{\text{u}} \frac{L'}{L'_{\text{u}}} \quad (3)$$

In the case when the unoccluded integral component can be computed for a particular light source in the single-scattering case, the unoccluded integral can be expressed as a sum of a stochastic integral and analytic unoccluded term ($L_{u,i}$),

$$L''_u(\omega_o,\omega_i) = L_s(\omega_i,\omega_o)\, e^{-\sigma_t t_{SC}} + + \alpha L_{u,i} \\ \beta \int_0^{t_{SC}} e^{-\sigma_t (t+d_m(t))} \sigma_s \int_{\mathbb{S}} f(\omega_i,\omega_o)\, L_i \,\mathrm{d}t,$$

where the weights (α, β) can be computed using multiple importance sampling [2].

3 Projected distance along the camera ray

Computation starts by considering a ray ($\tilde{\mathbf{p}}_1 + t_{\text{progress}}\, \tilde{\mathbf{w}}_c$) and a point $\tilde{\mathbf{p}}_2$ that must lie on the same light plane defined by its normal $\tilde{\mathbf{n}}_l$,

$$(\tilde{\mathbf{p}}_1 + t_{\text{progress}}\, \tilde{\mathbf{w}}_c) \cdot \tilde{\mathbf{n}}_l = \tilde{\mathbf{p}}_2 \cdot \tilde{\mathbf{n}}_l. \quad (4)$$

Rewriting the equation to find the distance along the ray,

$$t_{\text{progress}} = \frac{(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_1) \cdot \tilde{\mathbf{n}}_l}{\tilde{\mathbf{w}}_c \cdot \tilde{\mathbf{n}}_l}, \quad (5)$$

and substituting the distance vector between the two points ($\tilde{\mathbf{w}}_d = \tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_1$) yields the final result,

$$g_{\text{progress}}\big(\tilde{\mathbf{w}}_d, \tilde{\mathbf{w}}_c\big) = t_{\text{progress}} = \frac{\tilde{\mathbf{w}}_d \cdot \tilde{\mathbf{n}}_l}{\tilde{\mathbf{w}}_c \cdot \tilde{\mathbf{n}}_l}, \quad (6)$$

4 Integral of unoccluded radiance in a trapezoid segment

The transmittance is defined by distance to the vertex connecting the camera origin and the exit point from the medium along the incident light direction. The total distance is therefore a sum of two terms,

$$t_{\text{total}} = t_{C,L} + d_{C,L}. \quad (7)$$

The first distance is the integration variable in the currently derived integral ($t \equiv t_{C,L}$), while the second parameter can be derived as the linear interpolation between two distances. The first being the starting distance from the camera ray to the edge of the box ($d_{\text{E,S}}$),

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and the second being the distance from the end of the line segment to the edge of the box ($d_{\mathrm{E,E}}$),

$$d_{C,L} = d_{\mathrm{E,S}} + (t - t_{C,S}) \frac{d_{\mathrm{E,E}} - d_{\mathrm{E,S}}}{t_{C,E} - t_{C,S}}. \tag{8}$$

Plugging those values in the single-scattering integral equation results in the relation,

$$I_{\text{trapezoid}} = \int_{t_{C,S}}^{t_{C,B}} e^{-\sigma_t \left(t + d_{\mathrm{E,S}} + (t - t_{C,S}) \frac{d_{\mathrm{E,E}} - d_{\mathrm{E,S}}}{t_{C,E} - t_{C,S}} \right)} \,\mathrm{d}t, \tag{9}$$

where the integration domain covers the distance along the camera ray, starting from a distance $t_{C,S}$ and ending at the constrained distance ($t_{C,B} = \mathrm{clamp}(t_S, t_{C,S}, t_{C,E})$). That's a very well-known exponential function integral, which can be derived by rewriting the relation to expose a common constant factor,

$$\begin{aligned} I_{\text{trapezoid}} = {} & e^{-\sigma_t \left(d_{\mathrm{E,S}} - t_{C,S} \frac{d_{\mathrm{E,E}} - d_{\mathrm{E,S}}}{t_{C,E} - t_{C,S}} \right)} \cdot \\ & \int_{t_{C,S}}^{t_{C,B}} e^{-\sigma_t \left(1 + \frac{d_{\mathrm{E,E}} - d_{\mathrm{E,S}}}{t_{C,E} - t_{C,S}} \right) t} \,\mathrm{d}t. \end{aligned} \tag{10}$$

The slope can be substituted with a constant,

$$c = 1 + \frac{d_{\mathrm{E,E}} - d_{\mathrm{E,S}}}{t_{C,E} - t_{C,S}}, \tag{11}$$

which results in the equation,

$$\begin{aligned} I_{\text{trapezoid}} = {} & \frac{1}{\sigma_t c} e^{-\sigma_t \left(d_{\mathrm{E,S}} - t_{C,S} d_{\mathrm{E,E}} \frac{d_{\mathrm{E,E}} - d_{\mathrm{E,S}}}{t_{C,E} - t_{C,S}} \right)} \cdot \\ & \left(e^{-\sigma_t c t_{C,S}} - e^{-\sigma_t c t_{C,B}} \right), \end{aligned} \tag{12}$$

we can further extract the slope in the exponential factor in front of the parentheses,

$$\begin{aligned} I_{\text{trapezoid}} = {} & \frac{1}{\sigma_t c} e^{-\sigma_t \left(d_{\mathrm{E,S}} + t_{C,S} - t_{C,S} c \right)} \cdot \\ & \left(e^{-\sigma_t c t_{C,S}} - e^{-\sigma_t c t_{C,B}} \right). \end{aligned} \tag{13}$$

The final equation is derived after cancelling the common terms,

$$I_{\text{trapezoid}} = \frac{1}{\sigma_t c} e^{-\sigma_t (d_{\mathrm{E,S}} + t_{C,S})} \left(1 - e^{-\sigma_t c (t_{C,B} - t_{C,S})} \right). \tag{14}$$

In the main text [1] of the publication this equation is split into two separate terms,

$$\begin{aligned} E_u &= e^{-\sigma_t c (t_{C,B} - t_{C,S})} \\ E_l &= e^{-\sigma_t (d_{\mathrm{E,S}} + t_{C,S})} \\ I_{\text{trapezoid}} &= \frac{1}{\sigma_t c} (1 - E_u) E_l. \end{aligned} \tag{15}$$

However, this equation becomes singular when the slope approaches zero ($c \to 0$),

$$I_{\text{trapezoid}} = e^{-\sigma_t \left(d_{\text{edge},S} + t_{C,S} \right)} \int_{t_{C,S}}^{t_{C,B}} \lim_{c \to 0} e^{-\sigma_t c (t - t_{C,S})} \,\mathrm{d}t, \tag{16}$$

In this case the exponential function integrand approaches one and the whole integral is over a constant value,

$$\begin{aligned} I_{\text{trapezoid}} &= e^{-\sigma_t \left(d_{\text{edge},S} + t_{C,S} \right)} \int_{t_{C,S}}^{t_{C,B}} \mathrm{d}t \\ &= (t_{C,B} - t_{C,S}) E_l \end{aligned} \tag{17}$$

Both cases are combined to produce the final integral used in the main text,

$$\mathcal{T}_{\text{trapezoid}}(t_{C,S}, t_{C,E}, d_{\mathrm{E,S}}, d_{\mathrm{E,E}}).$$

5 Sampling according to unoccluded radiance in a trapezoid segment

The integral as expressed can be converted to cumulative density function by taking a variable distance boundary and normalizing it against its highest value at distance $t_{C,B}$,

$$\begin{aligned} F(t) &= \frac{\frac{1}{\sigma_t c} e^{-\sigma_t (d_{\text{edge},S} + t_{C,S})} \left(1 - e^{-\sigma_t c (t - t_{C,S})} \right)}{\frac{1}{\sigma_t c} e^{-\sigma_t (d_{\text{edge},S} + t_{C,S})} \left(1 - e^{-\sigma_t c (t_{C,B} - t_{C,S})} \right)} \\ &= \frac{1 - e^{-\sigma_t c (t - t_{C,S})}}{1 - e^{-\sigma_t c (t_{C,B} - t_{C,S})}}. \end{aligned} \tag{18}$$

The inversion method is performed by assigning a random value from uniform distribution and finding the inverse mapping,

$$\begin{aligned} r &= \frac{1 - e^{-\sigma_t c (t - t_{C,S})}}{1 - e^{-\sigma_t c (t_{C,B} - t_{C,S})}} \\ 1 - e^{-\sigma_t c (t - t_{C,S})} &= (1 - E_u) r \\ -\sigma_t c (t - t_{C,S}) &= \log(1 - (1 - E_u) r) \\ t &= t_{C,S} - \frac{1}{\sigma_t c} \log(1 - (1 - E_u) r). \end{aligned} \tag{19}$$

In the special case of the common term approaching zero ($c \to 0$), the sampling strategy can be also derived analytically,

$$\begin{aligned} r &= \frac{(t - t_{C,S}) E_l}{(t_{C,B} - t_{C,S}) E_l} = \frac{(t - t_{C,S})}{(t_{C,B} - t_{C,S})} \\ t &= t_{C,S} + (t_{C,B} - t_{C,S}) r. \end{aligned} \tag{20}$$

The final sampling algorithm is as expressed in the main text of the publication,

$$S_{\mathrm{s}}(r, \tilde{\mathbf{p}}, \tilde{\mathbf{w}}, t_{C,S}, t_{C,E}, d_{\mathrm{E,S}}).$$

6 Applying a circular gobo

A circular gobo limits the rays in an infinite cylinder. Given a point on the cylinder center axis $\mathbf{p}_g$, direction of the center axis $\mathbf{w}_g$ and radius of the gobo r_g, the intersection points with the gobo can be computed by solving the following quadratic equation,

$$\begin{aligned} \mathbf{p}_r &= \mathbf{p}_g - \mathbf{p}_o - ((\mathbf{p}_g - \mathbf{p}_o) \cdot \omega_i) \omega_i \\ \mathbf{w}_r &= -\omega_o + (\omega_o \cdot \omega_i) \omega_i \\ |\mathbf{p}_r + t \mathbf{w}_r|^2 - r_g^2 &= 0. \end{aligned} \tag{21}$$

The resulting distances are then limited to be in range of the camera ray $t'_{0,1} = \mathrm{clamp}(t_{0,1}, 0, \min(t_S, t_C))$. Afterwards, the lower boundary is used to offset the camera origin,

$$\mathbf{p}'_o = \mathbf{p}_o - t_0 \, \omega_o, \tag{22}$$

and computations are carried with this new camera origin. The final integral is attenuated by transmittance along the camera ray to reach that starting point,

$$\mathcal{T} = \mathcal{T}_{\text{box}}(\omega_i, \omega_o, \mathrm{clamp}(t_S - t_0, 0, t_1 - t_0)) \, e^{-\sigma_t t_0}, \tag{23}$$

where the distance to the surface is further constrained to lie within the gobo. When sampling the distance should be offsetted by the near distance to the gobo t_0.

The concepts outlined in this section extend to any analytic shape that can be represented on the plane. Custom shapes can be further made by applying the DDA algorithm [3] to find the extent of a shape represented by a mask image. Concave shapes will require splitting the integration into multiple segments.

7 Can you correct back from a bigger medium computation to a box?

Suppose you have a semi-infinite medium or a medium of different shape. Can you correct back from a technique targeting different medium and compute a mathematically correct result? Suppose that the medium can be split into trapezoids. We can solve the problem for a single segment and it should generalize to a complex medium section. The integral equation in the general case involves a sum of distances from the start ($d_{\mathrm{E,S,1}}, d_{\mathrm{E,S,2}}$) and end ($d_{\mathrm{E,E,1}}, d_{\mathrm{E,E,2}}$) of two media,

$$\begin{aligned} E'_l &= \exp(-\sigma_t\,(t_{C,S} + d_{\mathrm{E,S,1}} + d_{\mathrm{E,S,2}})) \\ E'_u &= \exp\,(-\sigma_t\,(t_{C,B} - t_{C,S})\cdot \\ &\quad \left(1 + \frac{d_{\mathrm{E,E,1}} - d_{\mathrm{E,S,1}}}{t_{C,E} - t_{C,S}} + \frac{d_{\mathrm{E,E,2}} - d_{\mathrm{E,S,2}}}{t_{C,E} - t_{C,S}}\right)\Bigg) \\ c' &= 1 + \frac{d_{\mathrm{E,E,1}} - d_{\mathrm{E,S,1}}}{t_{C,E} - t_{C,S}} + \frac{d_{\mathrm{E,E,2}} - d_{\mathrm{E,S,2}}}{t_{C,E} - t_{C,S}}. \end{aligned}$$

Each component can be separated according to the distance of each separate medium,

$$\begin{aligned} E_{l,k} &= \exp(-\sigma_t\,(t_{C,S} + d_{\mathrm{E,S,k}})) \\ E_{u,k} &= \exp\left(-\sigma_t\,(t_{C,B} - t_{C,S})\left(1 + \frac{d_{\mathrm{E,E,k}} - d_{\mathrm{E,S,k}}}{t_{C,E} - t_{C,S}}\right)\right) \\ c_k &= 1 + \frac{d_{\mathrm{E,E,k}} - d_{\mathrm{E,S,k}}}{t_{C,E} - t_{C,S}}, \end{aligned} \tag{24}$$

which allows substituting into the original equations,

$$\begin{aligned} E'_l &= \exp(\sigma_t t_{C,S})\,E_{l,1}\,E_{l,2} \\ E'_u &= \exp\,(\sigma_t\,(t_{C,B} - t_{C,S}))\,E_{u,1}\,E_{u,2} \end{aligned}$$

The integral equation for a single medium in the general case, ignoring the corner case, is defined based on the previously stated terms,

$$I_k = \frac{E_{l,k}(1 - E_{u,k})}{\sigma_t\,c_k}. \tag{25}$$

Then the complete integral can be combined by directly substituting the intermediate terms,

$$I' = \frac{\exp(\sigma_t t_{C,S})\,E_{l,1}\,E_{l,2}\,(1 - \exp\,(\sigma_t\,(t_{C,B} - t_{C,S}))\,E_{u,1}\,E_{u,2})}{\sigma_t\,c'}. \tag{26}$$

If we want to separate just the first term there is not a good way of doing it, the closest solution is

$$I' = \frac{\exp(\sigma_t t_{C,S})\,I_1\,E_{l,2}\,(1 - \exp\,(\sigma_t\,(t_{C,B} - t_{C,S}))\,E_{u,1}\,E_{u,2})\,c_1}{c'\,(1 - E_{u,1})}, \tag{27}$$

which turns out to be a highly complex way of deriving back the correct solution,

$$I_1 = I'\frac{c'\,(1 - E_{u,1})}{\exp(\sigma_t t_{C,S})\,E_{l,2}\,(1 - \exp\,(\sigma_t\,(t_{C,B} - t_{C,S}))\,E_{u,1}\,E_{u,2})c_1} \tag{28}$$

References

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[3] J. E. Bresenham, "Algorithm for computer control of a digital plotter," *IBM Syst. J.*, vol. 4, no. 1, pp. 25–30, Mar. 1965.